ISRAEL JOURNAL OF MATHEMATICS 98 (1997), 101–139

K-GROUPS ASSOCIATED WITH SUBSTITUTION MINIMAL SYSTEMS

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ABSTRACT

Two ordered Bratteli diagrams can be constructed from an aperiodic substitution minimal dynamical system. One, the proper diagram, has a single maximal path and a single minimal path and the Vershik map on the path space can be extended homeomorphically to a map conjugate to the substitution system. The other, the improper diagram, encodes the substitution more naturally but often has many maximal and minimal paths and no continuous compact dynamics. This paper connects the two diagrams by considering their K_0-groups, obtaining the equation

$$K_0(\text{Proper}) = K_0(\text{Improper})/Q \oplus \mathbb{Z}^\nu$$

where Q and ν can be determined from the combinatorial properties of the substitution. This allows several examples of substitution sequences to be distinguished at the level of strong orbit equivalence.

A final section shows that every dimension group with unit which is a stationary limit of $\mathbb{Z}^n$ groups can be represented as a K^0 group of some substitution minimal system. Also every stationary proper minimal ordered Bratteli diagram has a Vershik map which is either Kakutani equivalent to a d-adic system or is conjugate to a substitution minimal system.

The equation above applies to a much wider class which includes those minimal transformations which can be represented as a path-sequence dynamical system on a Bratteli diagram with a uniformly bounded number of vertices in each level.

Received August 7, 1994 and in revised form September 28, 1995

Introduction and definitions

The work of [HPS, GPS, S] shows that the orbit structure of Cantor dynamical systems can be classified by means of various dimension group invariants associated with the K_0 group of a certain C*-algebra. These in turn are closely connected with a combinatorial description of the dynamics developed first in [LV] and extended in [Pu, HPS, GPS, S]. The combined power of these two new methods promises to be of growing use in the study of Cantor dynamical systems.

The substitution minimal systems form an important class of dynamical systems and it is interesting to be able to determine their orbit equivalence classes conveniently. To this end, this paper characterises the dimension groups which can be produced as orbit invariants of a substitution minimal system. Further, this paper develops efficient ways of computing the group invariants (without order) for substitution minimal systems and a wide class of other subshifts and so to distinguish the orbit structure of these systems. As yet, however, the order structure of the invariants is incompletely described by the techniques in this paper.

The method developed here should be contrasted with Host's analysis [H] of the K_0 group invariants in which several constructions of the dimension group (complete with its unit and order) are compared. Host gives a particularly elegant construction in the case of a substitution system which is quite different from the one taken in this paper. Here there is more emphasis on the comparison of the groups obtained from the so-called proper diagram and the improper diagram.

In this first section, the notation and constructions from the various fields involved in this study will be gathered for use in later sections.

SUBSTITUTIONS. The following is a summary, without proof, of some of the definitions and results needed to start examining substitutions of non-constant length. See [D2, Q] for details.

Let $\Lambda = \{0, 1, \ldots, \lambda - 1\}$, $\lambda \geq 2$ be a finite set of letters and define $\Lambda^{(<\omega)}$ as the set of finite words.

This paper will consider substitutions, not necessarily of constant length: σ: $\Lambda \to \Lambda^{(<\omega)}$ one-to-one and whose values have length at least 2. (This excludes at first the usual substitution description of the Chacon example: $0 \to 0010, 1 \to 1$.)

σ can be considered as a map from Λ^n or $\Lambda^{\mathbb{N}}$ by making σ act coordinatewise with concatenation of the resulting finite sequences. In the latter case, $\sigma\colon \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ is a continuous map with respect to the metric

$$d(x,y) = \inf\{2^{-k}\colon x_i = y_i\ \forall 0 \le i \le k\}.$$

σ can act on $\Lambda^{\mathbb{Z}}$ keeping track of the zeroth coordinate in the following way: The notation *...ab.cde...* refers to a finite or infinite word β, whose zeroth coordinate, β_0 is c, $\beta_{-1} = b$ etc. Then $\sigma(\beta) = \cdots\sigma(a)\sigma(b).\sigma(c)\sigma(d)\cdots$ where the zeroth coordinate is the first letter of $\sigma(c)$ and the -1th coordinate is the last letter of $\sigma(b)$ etc. This is continuous with respect to the natural metric: $d(x,y) = \inf\{2^{-k}\colon x_i = y_i\ \forall 0 \le |i| \le k\}$.

Call a pair bc cyclic (see [D2] introduction) if there is a $k \ge 1$ such that the first letter of $\sigma^k(c)$ is c and the last letter of $\sigma^k(b)$ is b.

If bc is cyclic, then there is a unique bi-infinite string which is the limit of the sequence $\sigma^{kn}(b.c)$ as $n \to \infty$. Indeed, on the set $Z(bc) = \{x\colon x_0 = c, x_{-1} = b\}$ with complete metric d, σ^k is a contraction mapping with this limit as unique fixed point. Call this point $\omega(b.c)$.

A cyclic pair, bc, is called recurrent (as in [D2]) if the sequence $\omega(b.c)$ is uniformly recurrent in the sense of [F]. Equivalently: The topological dynamical system generated by $\omega(b.c)$ in $\Lambda^{\mathbb{Z}}$ with the shift is minimal. Equivalently again: If a appears in $\omega(b.c)$ then bc appears in $\sigma^{kn}(a)$ for some n large enough.

It is not hard to show that there is a recurrent pair for every substitution scheme as defined above.

Now fix a recurrent pair ai, and write $\omega = \omega(a.i)$ and construct the minimal shift system generated by ω : $(X(\omega), S)$ with distinguished point ω.

Throughout this paper it will be assumed that the sequence ω is not periodic. Periodic substitution sequences are quite easy to construct, for example $0 \to 010$, $1 \to 101$

Note that if bc is cyclic and bc appears in ω, then $\omega(b.c) \in X(\omega)$ and bc is recurrent therefore.

Example: The Morse-Thue substitution: $\sigma(0) = 01$, $\sigma(1) = 10$: has all pairs recurrent ($k = 2$) and $X(\omega)$ is independent of the choice of starting recurrent pair. This is a well-known example in topological dynamics [e.g. Ma, Ke, Pa] and Ergodic Theory [D1, K, Q] et al.

Having settled on $a.i$, Λ may be reduced, without loss of generality, to the set of only those letters which appear in ω. Also sequences, ω, which are eventually periodic are excluded from consideration.

These conditions and definitions all together define what is known usually as a primitive substitution. However, one further reduction is very useful; a consequence of the results of [M]: Without loss of generality it can be assumed that $\sigma^n\colon \Lambda \to \Lambda^{(<\omega)}$ is 1-1 for each choice of $n \geq 1$. This is in the sense that every primitive substitution sequence ω forms a shift system which is topologically conjugate to the shift system formed by a substitution sequence with this 1-1 property. Indeed, a proof of this fact can be extracted from the second half of the proof of Theorem 17 of this paper.

Definition: For the purposes of this paper, all the definitions, reductions and assumptions above constitute a primitive substitution.

It may be assumed, by replacing σ with a suitable power of σ, that, without loss of generality, $k = 1$ in all the definitions above and that, if b and c both appear in ω then b appears in $\sigma(c)$. Replacing σ with a higher power does not change the dynamics.

A finite or infinite word in which one of the letters is distinguished as the 0th. coordinate will be referred to as a zeroed word. A zeroed word that occurs in another zeroed word in the correct position will be referred to as a zeroed subword. Thus $a.b$ is a zeroed subword of $xya.bc$ but $x.y$ is not, although both are subwords.

The following constuctions are adapted from [M] which deals with the phenomenon of recognisability.

Let $\omega^+ = \lim_n \sigma^n(.i)$, the positive side of ω. Since $\omega^+ = \sigma(\omega^+)$ it is natural to define $B_1^+ \subset \mathbb{N}$, the collection of 'beats' in the sequence. More exactly: Let $|\alpha|$ be the length of a finite word α and write $\omega^+ = \omega_0\omega_1\omega_2\cdots$. Then $B_1^+ = \{\sum_{j=0}^{N-1} |\sigma(\omega_j)|\colon N = 1, 2, \ldots\}$.

Similarily, B_k^+, the k-beats in ω^+, can be defined

$$B_k^+ = \{\sum_{j=0}^{N-1} |\sigma^k(\omega_j)|\colon N = 1, 2, \ldots\}.$$

Also define $B_k = \{\sum_{j=0}^{N-1} |\sigma^k(\omega_j)|, -\sum_{j=-N}^{-1} |\sigma^k(\omega_j)|\colon N = 1, 2, \ldots\}$, the k-beats of ω.

Finite subwords of ω, ω^+ or other infinite words x are determined by their endpoints: Let

$$x[r,s] = x_r x_{r+1} \cdots x_{s-1}$$

a slight variation on the definition in [M]. Note that $x \in X(\omega)$ if and only if every finite subword of x is a finite subword of ω^+.

THEOREM 1 (Mossé [M]): *If ω is a primitive substitution sequence, then there is an L so that for each $t > L \in \mathbb{N}$, either*

(i) $\{s > L\colon \omega^+[s-L, s+L] = \omega^+[t-L, t+L]\} \subset B_1^+$ *or*

(ii) $\{s > L\colon \omega^+[s-L, s+L] = \omega^+[t-L, t+L]\} \cap B_1^+ = \emptyset$.

Thus every subword of ω of sufficient length occurs in ω^+ either always positioned 'on the beat' or always positioned 'off the beat'. The same theorem can be made for B_k^+ by an inductive argument and extended to cover B_k by minimality of the sequence. The following corollary expresses the generality that will be sufficient for later sections:

COROLLARY 2: *If ω is a primitive substitution sequence, and $M, k \geq 1$, then there is an $L = L(k, M)$ such that for each $u > L \in \mathbb{N}$ and each $0 \leq j \leq k$ and $|t| \leq M$, either*

(i) $\{s\colon \omega[s-L, s+L] = \omega[u-L, u+L]\} \subset B_j - t$ *or*

(ii) $\{s\colon \omega[s-L, s+L] = \omega[u-L, u+L]\} \cap B_j - t = \emptyset$.

The subtlety of this theorem and corollary lies mainly in the fact that in general nothing can be deduced about the beat structure at the edges of the subword, only at the centre. Examples that this is so are necessarily of non-constant length [M]. See [Ma] for the stronger recognisability condition that can be deduced in the constant length case.

BRATTELI DIAGRAMS. Again without proof, here are the definitions, constructions and references necessary to formulate the connection between dimension groups and orbit equivalence in symbolic dynamical systems. The basic combinatorial construction is essentially that of Livshitz and Vershik (see [LV]). This is presented here with an essential modification due to Herman, Putnam and Skau [HPS, S] who also established the connection with topological orbit equivalence.

A Bratteli diagram is an infinite graph with a partial order on its edges which is constructed in levels as follows:

The levels are disjoint finite sets of vertices V_n, with V_0 = single point. Edges, E_n, possibly multiple, are defined only between the vertices of consecutive levels V_n and V_{n+1}. The end-point of an edge $e \in E_n$ which lies in V_n is called the source of that edge and the end-point in V_{n+1}, the range. Each E_n is partially ordered so that edges with common range are comparible.

The complete assembly, $I = \bigcup(V_n, E_n)$, is called an ordered Bratteli diagram (see [HPS]).

A sequence of edges, one from each of E_n so that the source of the edge from E_n equals the range of the edge from E_{n-1}, is called a (infinite) path. Two paths agree up to level n if the edges picked in turn from $E_0 \cdots E_{n-1}$ of each path coincide. In the same way, finite paths can be defined between any two levels, or inifinte paths can be defined starting from any level. A path, p, passes through a vertex, v, at level n, if $v \in V_n$ and v is the source of the edge in p from E_n.

It is assumed that every vertex is connected to the initial point by a finite path of the kind defined above. Often it is required that every vertex in V_n is connected by at least one edge to each vertex in V_{n+1} and such a diagram is called 'minimal' or 'simple' [HPS].

Two infinite paths are cofinal if their edges coincide from some level on. Two such paths, p, p', are comparable in a partial order induced by the partial order on the edges as follows: Let $e_0, e_1, \dots e_{n-1}$ and $e'_0, e'_1, \dots e'_{n-1}$ be the initial segments of the two paths up to level n beyond which the paths coincide. Suppose that $e_{n-1} \neq e'_{n-1}$ and note that these edges have common range. Then $p < p'$ if $e_{n-1} < e'_{n-1}$.

A path is maximal if it is maximal with respect to this order. In this case each edge in this path is maximal among the other edges with the same range. A path can be maximal to a given level in this sense. Indeed, for every vertex, v, there is a path from the initial vertex in V_0 to v which is maximal — this fact will be used often in the constructions of the next few sections. These finite maximal paths need not be the initial segment of some infinite maximal path — a complication which will be noted later.

A similar definition is made for minimal paths.

Note that, by compactness, an ordered Bratteli diagram always has a maximal path and a minimal path. On the other hand, there may be many infinite maximal paths, although if the size of the levels are uniformly bounded then there are a finite number of infinite maximal paths. Similarly for minimal paths. Later in the paper, attention will be restricted to the case that there are a finite number of minimal and maximal infinite paths, so covering the case of uniformly bounded vertex sets.

The space of infinite paths, P, is compact with respect to a metric that compares initial segments — this is sometimes called the Markov Compactuum [LV]. The map, V, which sends a path to its immediate successor, is well-defined and continuous on the complement in P of the set of maximal paths. This will be referred as the Vershik map, after [GPS]. It is 1-1 and its image is the complement of the set of minimal paths.

Sometimes, for example if there is a single maximum and a single minimum, V can be extended homeomorphically to the whole of P, associating each maximal path to a minimal path as necessary. This homeomorphic 'glueing' cannot proceed in general, however, and remains a problem in many cases.

Connection matrices: $J^{(m,n)}$: $0 \leq n < m$ is the $|V_m| \times |V_n|$ matrix whose i, j entry records the number of edges (≥ 0) from vertex j of level n to vertex i of level m. Thus $J^{(n,m)} J^{(m,p)} = J^{(n,p)}$ etc. The Bratteli diagram can be encoded as the sequence of matrices, $J^{(1,0)}, J^{(2,1)}, J^{(3,2)}, \ldots$.

$J^{(m,n)}$ maps $\mathbb{Z}^{V_n} \to \mathbb{Z}^{V_m}$ and so define a sequence of positive group homomorphisms on which a direct limit can be taken: $K_0(I) = \lim_J \mathbb{Z}^{V_n}$. The representation of this that will be used here is the set $\bigcup_{n \in \mathbb{N}} \{n\} \times \mathbb{Z}^{V_n}$ quotiented by the relation $(n, v) \equiv (m, w)$, $m \geq n$, if $J^{(m,n)} v = w$. The process of addition quotients through this naturally: $[(n, v)] + [(m, w)] = [(p, z)]$ whenever there are representatives $(q, v'), (q, w')$ and (q, z') of each of these classes so that $v' + w' = z'$. The notion of positive element is also well defined: The class $[(n, v)] \geq 0$ if it has a representative, (n, v), so that all coordinates of v are ≥ 0. There is a natural order unit $\mathbb{I} = [(0, (1))]$. Thus $(K_0(I), \geq, \mathbb{I})$ is a dimension group with unit as in [HPS] et al.

Other order units in this K_0 may be specified giving a possibly non-isomorphic dimension group with unit.

Note that this construction does not depend upon the ordering on edges or paths.

An (ordered) Bratteli diagram is said to be stationary if V_n have the same number of elements for all $n > 0$ and there is an order on each V_n: $n > 0$ so that the graphs $(V_n \cup V_{n+1}, E_n)$: $n > 0$, taking into account the order on the vertices (and on the edges in the case of ordered diagrams), are identical. This appears in the picture itself as a repetition, between every level of vertices, of the graph found between the second two levels. The list of connection matrices which correspond to such a diagram will therefore be constant from the second term on.

DYNAMICS. (See [P, W] for a general introduction.) Suppose that X is a Cantor set (compact metric, totally disconnected without isolated points) and that $T: X \to X$ is a homeomorphism onto. The combination, (X, T), is called a Cantor dynamical system. In this paper, attention will be restricted to the case that $X \subset A^{\mathbb{Z}}$, where A is a finite set of letters, and T is the bilateral shift, S, i.e. subshifts. Many arguments proceed for the general case, however.

(X, T) is said to be minimal if there is no proper closed subset F so that $TF = F$. Equivalently, every orbit, $O(x) = \{T^n x: n \in \mathbb{Z}\}$, is dense. Equivalently again: if Z is a non-empty clopen subset of X, then $n(x) = \min\{n \geq 1: T^n x \in Z\}$ is well-defined and continuous on all of X.

In this latter case, an induced homeomorphism can be defined on Z: $T_Z: Z \to Z$ where $T_Z x = T^{n(x)} x$. Thus (Z, T_Z) becomes a dynamical system which can be shown to be minimal subject to the conditions above.

Two systems, $(X, T), (Y, U)$, are orbit equivalent if there is a homeomorphism $\phi: X \to Y$ such that single orbits $O(x)$ are mapped onto single orbits $O(y)$. Therefore minimality is a property preserved by orbit equivalence.

When there is orbit equivalence, there are maps $n: X \to \mathbb{Z}$ and $m: Y \to \mathbb{Z}$ so that if $y = \phi(x)$, then $\phi(Tx) = U^{n(x)} y$ and $\phi^{-1}(Uy) = T^{m(y)} x$. If both n, m are continuous except at a single point, then this is called strong orbit equivalence: a strictly stronger notion.

Two systems are strongly Kakutani orbit equivalent if they each have some induced system and these are strongly orbit equivalent.

Given B a sub $\mathbb{Z}$-module of $C(X: \mathbb{Z})$, write $\operatorname{cbdy}_S(B) = \{f - f \circ S: f \in B\}$. If B is S-invariant then $\operatorname{cbdy}_S(B) \subseteq B$.

Define $K^0(X, S) = C(X: \mathbb{Z}) / \operatorname{cbdy}_S(C(X: \mathbb{Z}))$. This is often written as $H^1(X, S, \mathbb{Z})$ in the literature [H] and this is indeed the same as a first group cohomology of $\mathbb{Z}$ and as the first Čech cohomology of a mapping torus. The notation used here is taken from [Pu, HPS] owing to the close connection with K-theory established in those papers.

Note that this group comes equipped with a natural order ($[f] \geq [g]$ if and only if there are $f \in [f]$ and $g \in [g]$ such that $f \geq g$ as elements of $C(X: \mathbb{Z})$) and a natural order unit (the equivalence class that contains the constant 1 function). This satisfies the axioms for dimension group with unit (see [HPS]). However, order properties will only be touched in the last section of this paper.

Here is the fundamental theorem of [GPS] which connects the properties of orbit equivalence to the K^0 groups.

THEOREM 3: *Suppose that all is set up as before, then (X,T) is strongly Kakutani orbit equivalent to (Y,U) if and only if $K^0(X,T)$ is order-isomorphic to $K^0(Y,U)$ as dimension group.*

So the orbit structure of a dynamical system is determined in part by the K^0 groups and it is important therefore to find an effective way of computing these. The work of [HPS] establishes a fruitful connection between K_0 and K^0 and this is outlined here for the case of subshifts. The details are important in this case since this construction will be referred to in Theorem 5.

THE PROPER DIAGRAM. Suppose that (X,S) is a minimal invertible aperiodic subshift and $\omega \in X$. The following procedure constructs from this an ordered Bratteli diagram:

Suppose that $Z_0 = X, Z_1, \ldots$ is a strict descending sequence of clopen sets such that $\bigcap Z_n = \{\omega\}$. Let $R_n = \{m \in \mathbb{Z}: S^m\omega \in Z_n\}$ so that $\{0\} = \bigcap_n R_n$, and $R_n \supset R_{n+1}$, for all n.

Write $m << m' \in R_n$ if $m, m' \in R_n$ and $m < m'$ have no other elements of R_n between them. For such a pair record the following information: The word $\omega[m,m']$: The sets $R_j[m,m']$ for $0 \leq j < n$ where

$$R_j[m,m'] = \{p - m: p \in R_j: m \leq p \leq m'\}.$$

The minimality of the system ensures that, for fixed $n \geq 0$, there is a uniform bound on $m' - m$ and so the information has a finite number of encodings as above and

$$V_{n+1} = \{(\omega[m,m'], R_0[m,m'], \ldots, R_{n-1}[m,m']): m << m' \in R_n\}$$

is finite therefore. The elements of V_1 will be in 1-1 correspondence with the alphabet A, for example.

These form the levels of vertices in an ordered Bratteli diagram which is constructed as follows:

Let $V_0 = \{$ single vertex $\}$ and connect each vertex in V_1 to V_0 by a single edge; there is no need to worry about the ordering on these edges.

For each $n \geq 2$, decompose each element, α, of V_n as an ordered list of elements from V_{n-1}: $\alpha = \beta_1\beta_2\cdots\beta_k$. In detail, if α is constructed from $m << m' \in R_{n-1}$ as above, and if $R_{n-2}[m, m']$ is written $\{0 = p_0 < p_1 < \cdots < p_{k-1} < p_k = m' - m\}$, then set

$$\beta_j = (\omega[m + p_{j-1}, m + p_j], R_0[m + p_{j-1}, m + p_j], \ldots, R_{n-3}[m + p_{j-1}, m + p_j]).$$

In the diagram, α is connected to $\beta \in V_{n-1}$ by an edge of ordinality j if $\beta = \beta_j$ in this decomposition.

This may be repeated for each n independently to construct a Bratteli diagram in which the initial vertex is connected to every other vertex.

To find which path corresponds to a particular element $x \in X$: Fix $n \geq 0$. Let

$$R_n(x) = \{m \in \mathbb{Z}\colon S^m x \in Z_n\}$$

and let $m_n(x) = \max\{m \leq 0\colon m \in R_n(x)\}$ and $m'_n(x) = \min\{m > 0\colon m \in R_n(x)\}$. Let

$$R_{j,n}(x) = \{p - m_n(x)\colon p \in R_j(x)\colon m_n(x) \leq p < m'_n(x)\}$$

defined for $0 \leq j < n$, and note that, by construction,

$$\alpha_n = (x[m_n(x), m'_n(x)], R_{0,n}(x), \ldots, R_{n-1,n}(x))$$

is in V_{n+1} and this forms the sequence of vertices that the path will go through. To determine the edge which the path takes between $\alpha_{n-1} \in V_n$ and $\alpha_n \in V_{n+1}$, write $R_{n-1,n} = \{0 = p_0 < p_1 < \cdots < p_{k-1} < p_k = m'_n(x) - m_n(x)\}$ and note that there is a unique j: $0 < j \leq k$ so that $m_n(x) + p_{j-1} = m_{n-1}(x)$ and $m_n(x) + p_j = m'_{n-1}(x)$. Therefore choose the edge with ordinality j. Continue in this way for every level and so build a path in the diagram constructed before.

This is the subshift version of the construction of [Pu] and [HPS, part 4], but it ignores the complication of whether further refinements need to be made by partitions on X which generate the topology. In fact, in the case of subshifts, no such complications arise since the construction of paths above distinguishes points.

By [HPS], this Bratteli diagram, J, has a unique maximum path and a unique minimum path, and the Vershik map, V, can be defined homeomorphically on the path space, $P(J)$.

The following connects the dynamics with the diagram and its K_0 group.

THEOREM 4 ([HPS]): *If (X, S) is a minimal Cantor system and J a Bratteli diagram derived from (X, S) by the procedure above, then*

(i) *$K^0(X, S) \equiv K_0(J)$ as groups (indeed as dimension groups with unit).*

(ii) *(X, S) is topologically conjugate to $(P(J), V)$.*

Bratteli diagrams associated with substitutions

Vershik and Livshitz [LV] construct an important example of an ordered stationary Bratteli diagram connected with a substitution scheme in the following natural way:

THE IMPROPER DIAGRAM. Suppose that σ: $\Lambda \rightarrow \Lambda^{(<\omega)}$ is a substitution scheme. Consider the stationary diagram each of whose vertex layers (beyond the initial single point) is a copy of Λ and whose periodic edge arrangement is formed by connecting $(n+1, \lambda)$ (i.e. the symbol λ in the $n+1$st. layer) to (n, λ') $(n \geq 1)$ by an edge with ordinal k whenever $\sigma(\lambda)_k = \lambda'$. The single point in the 0th. layer is connected by a single edge to each of the points of the 1st. layer.

For example, the repeating unit of the diagram associated to the Morse-Thue substitution is a graph of two two-point layers with all possible connections between the layers and no multiple edges (i.e. $K_{2,2}$ the complete bipartite graph on $2+2$ vertices). The ordinal assignment to edges is such that, if the points are arranged naturally in a square, then the diagonal edges have ordinal 1 and the lateral edges ordinal 0.

The Vershik map for the diagram in this example is not everywhere defined. There are two maximum paths and two minimum paths and there is no way of assigning each maximum to a unique minimum to obtain a continuous extension of the Vershik map defined as before. [LV] shows, however, that by ignoring the forward orbits of the minimum paths and the reverse orbits of the maximum paths (which are well-defined for the Vershik map), the continuity of the dynamics can be restored. Although not compact, the resulting system is uniquely ergodic and is metrically isomorphic to the uniquely ergodic substitution minimal system. So a correspondence in the category of measure-preserving dynamics is obtained.

The discussion above leaves the question of topological conjugacy unanswered, however. Indeed, there is no hope of a simple correspondence in this case since the Vershik map cannot even be defined everywhere and there is no second compact topological dynamical system to compare.

But all is not lost since the original substitution sequence can be retrieved from the forward orbit of a minimal path. Consider, the path, q^*, which passes through the points (i, n) at each level is minimal (recall $\sigma(i)_0 = i$, by assumption) and the map π which sends the path q to the letter at level 1 which lies on q. The sequence $(\pi(V^n q^*)\colon n \geq 0)$ is precisely the right-hand limit $\lim \sigma^n(.i)$.

Similarly, if p^* is the (maximal) path which passes through the points (a, n), then $(\pi(V^n p^*)\colon n \leq 0)$ is the limit $\lim \sigma^n(a.)$. Thus

$$\omega = \cdots \pi(V^{-1}p^*)\pi(p^*).\pi(q^*)\pi(Vq^*)\cdots.$$

The arguments of this section show that it is precisely the existence of many maxima and minima and their relationship which modifies the correspondence between the substitution minimal system and the dynamics of the diagram constructed by Vershik and Livshitz. Further, this modification can be made exact in terms of the K-groups.

The invertible minimal dynamical system, $(X(\omega), S)$, constructed from a minimal substitution scheme has a privileged point, namely the bi-sequence ω starting with $a.i$ and stationary with respect to application of σ. Thus a Bratteli diagram can be constructed from this using the general method of [HPS] above and it will not in general be the same as the one constructed by Vershik and Livshitz since it will have a well-defined Vershik map and unique maximum and minimum paths.

Definition: This latter diagram will be called a proper Bratteli diagram for the substitution.

The first construction of Vershik and Livshitz will be called the improper Bratteli diagram for the substitution.

Some technical notes. Recall the assumption on the substitution: that if b and c both appear in ω then b appears in $\sigma(c)$. This translates to the fact that, in the Improper diagram, every point in V_n is connected to every point of V_{n+1} by at least one edge in E_n — i.e. minimality of the diagram. (Regarding the over-use of the word minimal to refer to paths, dynamical systems and diagrams: the last two are almost the same and the first is quite distinct.)

A direct consequence of the minimality of the diagram: If p, q are any two paths and $t \geq 1$ is given, then there is a j such that $V^j p$ is well-defined and agrees with q to level t. To see this, take q to level t and continue it by some edge (which exists by minimality) to $p(t+1)$ on level $t+1$ and continue this arbitrarily to form a path, q', which agrees with q to level t. $q'(t+1) = p(t+1)$ and so the initial paths of p and q to level t are comparable and so there is a j such that $V^j p$ agrees with q' to level $t+1$ (j will be positive if the initial path for p is less than the initial path for q', negative otherwise). Thus $V^j p$ agrees with q to level t as required.

A maximal path and a minimal path do not agree beyond a certain level.

If p is minimal, then $V^n p$ is well-defined for all $n \geq 0$ and no $V^n p$: $n > 0$ is maximal or minimal. This is because if p is non-maximal, then Vp is well-defined and differs from p only in a finite number of edges. If $V^n p$ were maximal, then there would be a maximal path which would coincide with a minimal path beyond a certain level — a contradiction.

The main theorem of this section shows that the phenomenon of stationarity continues in the case of Proper Bratteli diagrams for substitution minimal systems:

THEOREM 5: *Suppose that σ is a primitive substitution scheme and $\omega = \omega(a.i)$ is the substitution sequence obtained from it as above; then there is a Proper Bratteli diagram for $(X(\omega), S, \omega)$ which is stationary.*

Proof: Recall the corollary of Mossé's Theorem from before (Corollary 2) and the sets B_k and bounds $L(k, M)$ produced there. Let $\lambda'_k = |\sigma^k(a)|$ and $\lambda_k = |\sigma^k(i)|$ and set $M_k = \max\{\lambda_k, \lambda'_k\}$ and $L_k = L(k, M_k)$.

Say that α, a subword of ω of length $2L_k$, has condition C_k if, for $-\lambda'_k \leq t \leq \lambda_k$ and $1 \leq j \leq k$, the set $\{s: \omega[s - L_k, s + L_k] = \alpha\} \subset B_j - t$ if and only if $t \in B_j$.

Note that, by Corollary 2, $\{s: \omega[s - L_k, s + L_k] = \alpha\} \subset B_j - t$, fails for $|t| \leq M_k$ if and only if $\{s: \omega[s - L_k, s + L_k] = \alpha\} \cap B_j - t = \emptyset$. Thus the condition C_k can be confirmed for α by examining any occurrence of α in ω.

Define $X_k = \{x \in X(\omega): x[-L_k, L_k]$ has condition $C_k\}$. This is a set defined by what happens on a finite number of coordinates and so it is clopen. Further, $\omega \in X_k$ for all k.

Let $Z_1 = \{x \in X(\omega): x_{-1} = a, x_0 = i\}$ and $Z_{k+1} = \{x \in X_k: x[-\lambda'_k, \lambda_k] = \sigma^k(ai)\}$ for $k \geq 1$. These are clopen and $\bigcap_k Z_k = \{\omega\}$, confirming the requirements for the construction of a proper ordered Bratteli diagram outlined in a previous section. Recall that $R_n = \{m \in \mathbb{Z}\colon S^m\omega \in Z_n\}$.

Proceed with the construction of the proper diagram and note the most important property that for $k \geq 0$, $0 \leq j \leq k$ and $-\lambda'_k \leq t \leq \lambda_k$, either $R_{j+1} \subset (B_j - t)$ (if and only if $t \in B_j$) or $R_{j+1} \cap (B_j - t) = \emptyset$ (otherwise).

This can be turned into a characterisation of elements of R_n: $r \in R_n$ if and only if

$$\omega[r - \lambda'_{n-1}, r + \lambda_{n-1}] = \sigma^{n-1}(ai)$$

and

$$\{-\lambda'_{n-1} \leq t \leq \lambda_{n-1}\colon r + t \in B_j\} = \{-\lambda'_{n-1} \leq t \leq \lambda_{n-1}\colon t \in B_j\}$$

for all $0 \leq j \leq n-1$.

Applying σ once to ω leaves ω unchanged, but subwords are substituted and moved outwards and their new position can be followed. In detail, σ defines a map τ from $\mathbb{Z}$ into $\mathbb{Z}$ as follows:

$$\tau(m) = \begin{cases} \displaystyle\sum_{0 \leq j \leq m-1} |\sigma(\omega_j)| & m \geq 0, \text{ and} \\ \displaystyle -\sum_{m \leq j \leq -1} |\sigma(\omega_j)| & \text{otherwise.} \end{cases}$$

This has the immediate properties: $\sigma(\omega[r,s]) = \omega[\tau(r), \tau(s)]$ and $B_{n+1} = \tau(B_n)$. It is less immediate but important for the continuing argument to show that $\tau(R_n) = R_{n+1}$ for all $n \geq 1$; the argument needed is given in the next three paragraphs:

Suppose that $r \in R_n$, so that it obeys the characterisation of R_n above. By applying σ to the first part of the characterisation, it is immediate that

$$\omega[\tau(r) - \lambda'_n, \tau(r) + \lambda_n] = \sigma^n(ai)$$

confirming the first part of the characterisation of R_{n+1}. Further, since

$$\omega[r - \lambda'_{n-1}, r + \lambda_{n-1}] = \sigma^{n-1}(ai)$$

then

$$\sum_{m=r}^{r+t-1} |\sigma(\omega_m)| = \sum_{m=0}^{t-1} |\sigma(\omega_m)| = \tau(t)$$

for all $0 \leq t \leq \lambda_{n-1}$, and similarily for $-\lambda'_{n-1} \leq t \leq 0$. This implies that $\tau(r+t) = \tau(r) + \tau(t)$ under these conditions. Thus

$$\begin{aligned} &\tau\{t\colon -\lambda'_{n-1} \leq t \leq \lambda_{n-1} \text{ and} r + t \in B_j\} \\ &= \{\tau(t)\colon -\lambda'_n \leq \tau(t) \leq \lambda_{n-1} \text{ and } \tau(r+t) \in B_{j+1}\} \\ &= \{t\colon -\lambda'_n \leq t \leq \lambda_{n-1} \text{ and } \tau(r) + t \in B_{j+1}\} \end{aligned}$$

and so the second condition for membership of R_{n+1} is obeyed, proving that $\tau(r) \in R_{n+1}$.

To show that τ maps from R_n onto R_{n+1}, suppose that $s \in R_{n+1}$ and use the characterisation above again. This shows that sitting in ω, the word $\omega[s-\lambda'_n, s+\lambda_n]$ is split by the beats B_1 exactly as if it were the word $\omega[-\lambda'_n, \lambda_n] = \sigma^n(a.i)$. Thus $\omega[s-\lambda'_n, s+\lambda_n] = \sigma(\omega[r-\lambda'_{n-1}, r+\lambda_{n-1}])$ for some r and $\tau(r) = s$. It is now necessary to show that this r comes from R_n. However, this is almost immediate by extending the argument above to higher levels of beats. Observe that, for each $1 \leq k \leq n$, the beats B_k break the word $\omega[s-\lambda'_n, s+\lambda_n]$ just as they break the word $\omega[-\lambda'_n, \lambda_n] = \sigma^n(a.i)$ so confirming the remaining conditions needed to put r in R_n.

Since τ is 1-1 in any case, it is therefore a bijection between R_n and R_{n+1} for any n.

More generally, a function

$$\tau(x, n) = \sum_{j=0}^{n-1} |\sigma(x_j)|$$

can be defined for $n \geq 0$ so long as $x_0 x_1 \cdots x_{n-1}$ are defined.

Fix $n \geq 1$ and recall that each element of V_{n+1} is defined by a list of information obtained from a pair, $r << s$, of consecutive elements of R_n. In this case, the list is

$$(\omega[r,s], R_0[r,s], \ldots, R_{n-1}[r,s]).$$

Let α be the finite word $\alpha_0 \cdots \alpha_{s-r-1} = \omega[r,s]$.

Since τ is 1-1 and onto R_{n+1}, the general consecutive pair in R_{n+1} is of the form $\tau(r) = r' << \tau(s) = s' \in R_{n+1}$. The list which corresponds to this pair is

$$(\omega[r',s'], R_0[r',s'], \ldots, R_n[r',s'])$$

and, by construction,

$$\omega[r',s'] = \sigma(\alpha),$$

$$R_0[r', s'] = \{0, \ldots, s' - r'\}$$

and

$$R_j[r', s'] = \tau(\alpha, R_{j-1}[r, s])$$

(i.e. the image of $R_{j-1}[r, s]$ under the map $\tau(\alpha, .)$) for all $1 \leq j \leq n$. So the function

$$\Sigma(\alpha, A_0, \ldots, A_n) = (\sigma(\alpha), \{0, 1, \ldots, |\sigma(\alpha)|\}, \tau(\alpha, A_0), \ldots, \tau(\alpha, A_n))$$

is a map from V_n onto V_{n+1}.

Recall the construction of edges in the proper diagram and note that the map Σ respects these connections. I.e. $w \in V_{n+1}$ is connected to $v \in V_n$ by an edge with ordinality j, then $\Sigma(w)$ is connected to $\Sigma(v)$ by an edge with ordinality j.

Since all the V_n are finite and the Σ onto, Σ: $V_n \to V_{n+1}$ must be 1-1 for all n large enough, hence bijective. For such n, Σ defines, therefore, an ordered graph isomorphism between $(V_n \cup V_{n+1}, E_n)$ and $(V_{n+1} \cup V_{n+2}, E_{n+1})$ which gives the stationarity required of the diagram at all levels high enough.

Also, the first few levels can be telescoped (see [HPS]) into one to get a true stationary diagram and the theorem is proved therefore.

Example: The Proper Bratteli diagram obtained from the Morse-Thue sequence using the method of the proof above has a periodic unit from the second level which involves 4 vertices at each level.

This has a connection matrix

$$J = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

This produces $K_0 = \mathbb{Z}[1/2] \oplus \mathbb{Z}$ where $\mathbb{Z}[1/2]$ is the group of diadic rationals.

This implies that the substitution system generated by the Morse-Thue sequence is not strongly Kakutani orbit equivalent to the 2-adic odometer. Thus an important distinction among dynamical systems can be deduced by inspection of the K_0 group.

Even in this example however, it is not clear how to connect the Proper Bratteli diagram by purely combinatorial means to the simpler Improper Bratteli diagram. In this paper, the connection is made via the K-groups involved. At first, the work will be applicable to Bratteli diagrams in general with a few light assumptions.

Path-sequence dynamical systems and their dimension group

Assume now that $I = (V_n, E_n)$ is a Bratteli diagram and that there are only a finite number of maximal and minimal paths, sets denoted M and N respectively. This will certainly occur if I is the improper diagram of a substitution sequence.

Without loss of generality, it may be assumed that the diagram is minimal in the sense that each element of V_n is connected by at least one edge to each element of V_{n+1}, for all $n \geq 0$. Also the initial edges of each maximal or minimal path on the first level can be assumed to be distinct. Thus pairs of maximal paths have no edges in common, nor do pairs of minimal paths and pairs minimal and maximal. All these properties can be assured by means of suitable 'telescoping' and 'microscoping' (see [HPS]); changes which do not affect the generality of the theorems to be proved.

Let P be the path space, compact with respect to the natural topology of initial agreement of path edges. The Vershik map V is defined on $P \setminus M$ and its inverse on $P \setminus N$.

Form a relation $\sim$ between M and N by connecting $p \in M$ to $q \in N$ if and only if there are paths $a_n \to p$ such that $Va_n \to q$ and paths $b_n \to q$ such that $V^{-1}b_n \to p$.

Note that every $p \in M$ is related to some $q \in N$ and every $q \in N$ is related to some $p \in M$. Also if $p \sim q$, then both p and q are unrelated to any other paths if and only if V may be extended homeomorphically by defining $Vp = q$.

Construct the following space:

$$\Pi = \{(p_n) \in P^{\mathbb{Z}}\colon \forall n \in \mathbb{Z} \text{ either } (p_{n+1} = Vp_n)$$
$$\text{or } (p_n \in M, p_{n+1} \in N \text{ and } p_n \sim p_{n+1})\}.$$

The shift, $S\colon P^{\mathbb{Z}} \to P^{\mathbb{Z}}$, is defined: $(S(p_n\colon n \in \mathbb{Z}))_m = p_{m+1}$.

It is straightforward to confirm that Π is compact as a subspace of $P^{\mathbb{Z}}$ with the Tychonov topology and S is a homeomorphism of Π onto itself. Further, Π has no isolated points and so is a Cantor set. The density of all orbits in Π can be deduced directly from the minimality of the Bratteli diagram and so (Π, S) is minimal. The system (Π, S) will be called the path-sequence space.

Given $p \sim q$, the point $x(p, q) = (p_n)$ is defined uniquely in Π by the condition $p_{-1} = p$ and $p_0 = q$.

For each $t \in \mathbb{N}$ and $p \in P$, consider the following function defined on $P^{\mathbb{Z}}$: $F_p^{(t)}((q_n)) = 1$ if the edges of q_0 agree with those of p up to level t, and $F_p^{(t)}((q_n)) = 0$ otherwise.

These are the indicators of basic clopen sets depending only on the zero coordinate. Thus the $F_p^{(t)}$ are continuous and the algebra generated by them and their translates by S (and without completion) is equal to $C(P^{\mathbb{Z}}\colon \mathbb{Z})$, the continuous $\mathbb{Z}$-valued functions on $P^{\mathbb{Z}}$. This algebra projects by restriction onto $C(\Pi\colon \mathbb{Z})$.

SOME GRAPH THEORY. Suppose that $G = (V, E)$ is a finite graph with simple edges and no loops. The edge-vertex matrix, $A(G)$, is defined to be the $0-1$ valued $|E| \times |V|$ matrix, columns indexed by vertices, $v \in V$, rows indexed by edges, $e \in E$, such that $A_{e,v} = 1$ iff $v \in e$, i.e. v is an end point of e. A can be thought of as a group homomorphism and $\mathbb{Z}$-module map on column vectors $A\colon \mathbb{Z}^V \to \mathbb{Z}^E$.

LEMMA 6: *Suppose that G is bipartite with c connected components, then*

$$\mathbb{Z}^E / \operatorname{Im}(A) = \mathbb{Z}^\nu$$

where $\nu = |E| - |V| + c$, and so $\operatorname{Im}(A)$ *is complemented in $\mathbb{Z}^E$.*

In fact, a complimentary subspace, $C \subset \mathbb{Z}^E$, can be constructed which has a basis of unit vectors indexed by a set of edges, E', whose removal from E leaves G cycle-free without increasing the number of components.

Proof: To show the existence of $\nu \geq 0$, it is sufficient to show that $\mathbb{Z}^E / \operatorname{Im}(A)$ is torsion free. Suppose, therefore, that $v \in \mathbb{Z}^E$, and $k > 1$ such that $kv \in \operatorname{Im}(A)$; it is sufficient to show that $v \in \operatorname{Im}(A)$.

The operator A may be considered combinatorially as a way of transferring an integer weight distribution on the vertices to an integer weight distribution on the edges by assigning to each edge the combined weights of its end-points. The fact that $kv \in \operatorname{Im}(A)$ shows that there is a distribution w on the vertices so that $A(w) = kv$. Thus the values of w to be found at either end of an edge sum to 0 mod k.

In the case of a connected bipartite graph this shows that w has at most two values $\pm t$ mod k, and the split coincides with the bipartition.

Pick a representative of this t from $\mathbb{Z}$ so that $0 \leq t < k$ and consider the weight distribution w' which equals t whenever w equals $-t$ mod k, and which equals $-t$ when w equals t mod k. Then $A(w') = 0$, $A(w - w') = kv$. Further, $w - w' = 0$ mod k and so $v = A((w - w')/k) \in \operatorname{Im}(A)$ as required.

For a disconnected graph, this construction can be made on each component independently to the same effect.

The complementation now follows from simple algebraic considerations.

The exact value of ν is obtained by dimensional considerations: The smallest complex vector-subspace of $\mathbb{C}^{|E|}$ which contains $\mathrm{Im}(A)$ is also the cut space (or coboundary space) of G (see [B], p.36ff. for example) which has dimension $|V|-c$. The complement therefore has dimension $|E|-|V|+c$ as required.

The construction of a complementary space can be made by considering a second meaning of the value of ν found in [B]. ν is the minimal number of edges which can be deleted from G to make it cycle-free. Let E' be a minimal set of edges to be deleted. Then the unit vectors in $\mathbb{Z}^E$ indexed by elements of E' span a space, C, complementary to $\mathrm{Im}(A)$ as required. The information about the number of components comes from considering the minimality of E'.

Finally, the definition of the subgroup, Q, of $K_0(I)$ where I is an ordered Bratteli diagram:

Recall the connection matrices, $J^{(t,s)}$, of the Bratteli diagram which count the number of paths from points in level s to points in level t. Introduce the vectors e_v, v a vertex in level t, which form the unit basis for $\mathbb{Z}^{V_t}$.

Recall also that an element of $K_0(I)$ is represented as an equivalence class of pairs $[(s,\alpha)]$ where $\alpha \in \mathbb{Z}^{V_s}$. $(s,\alpha) \equiv (t,\alpha'), t > s$ whenever $J^{(t,s)}\alpha = \alpha'$ etc.

Fix a level t. Suppose that $k > 0$, that $q = (q_n)$ is some path-sequence in Π and that, for each $v \in V_t$, p_v is some choice of infinite path which passes through vertex v at level t and is maximal to level t. This defines a vector $\sum_v \sum_{0\le m\le k} F^{(t)}_{p_v}(S^m q)e_v$. Let $\mathbb{B}_t \subset \mathbb{Z}^{V_t}$ be the set of all vectors that can be formed in this way.

Define $Q_t = \{\alpha \in \mathbb{Z}^{V_t}\colon \sup_{\beta\in\mathbb{B}_t} |\langle\beta,\alpha\rangle| < \infty\}$ where $\langle .,. \rangle$ is the usual euclidean inner product.

LEMMA 7: *For $s > t$, $J^{(s,t)}Q_t \subset Q_s$.*

Proof: Suppose that $\alpha \in Q_t$ with $\sup_{\beta\in\mathbb{B}_t} \langle\beta,\alpha\rangle = K$ and set $J = J^{(t+1,t)}$. Suppose that for each $w \in V_t$, p'_w is a path through w at level t and maximal to level $t+1$.

Splitting $F^{(t)}_{p_v}$ to the next level, $F^{(t)}_{p_v} = \sum_h F^{(t+1)}_{p_{h,v}}$ where the sum is over edges $h \in E_t$ whose source is v and where $p_{h,v}$ are paths coincident with p_v to level t and running along edge h to level $t+1$, thereafter proceeding arbitrarily. If $w \in V_{t+1}$ is the range of $h \in E_t$, then $|\sum_{0\le m\le k}(F^{(t+1)}_{p_{h,v}} - F^{(t+1)}_{p'_w})(S^m q)| \le 2$ uniformly in k, since $p_{h,v}$ and p'_w have a common vertex at level $t+1$.

Summing over h with common source v produces

$$\left|\sum_{0\le m\le k}(F^{(t)}_{p_v}-\sum_{w\in V_{t+1}}J_{w,v}F^{(t+1)}_{p'_w})(S^m q)\right|\le 2|E_t|$$

uniformly in k. Therefore $|\langle\beta', J\alpha\rangle| \le K+2|E_t|\max_v|\alpha_v|$ for all $\beta'\in\mathbb{B}_{t+1}$ as required for the lemma.

Definition: $Q=\{(t,\alpha)\colon \alpha\in Q_t\}$.

It is immediate that this is a subgroup defined by completely positive conditions and so $K_0(I)/Q$, being torsion free, is a well-defined dimension group with a unit inherited from $K_0(I)$.

Some further properties of Q which sometimes help its calculation:

LEMMA 8:
(i) *All $\alpha\in Q_t$ have the property that $\sup_s\|J^{(s,t)}\alpha\|\le K$ (same K as the definition) where the norm is l_∞ norm in $\mathbb{Z}^{V_s}$.*
(ii) *Q is a subgroup of the infinitesimals in $K_0(I)$ (see [GPS] for a definition).*
(iii) *$Q\equiv\mathbb{Z}^\mu$ for some μ.*

Proof: (i) The rows of the matrix $J^{(s,t)}$ are vectors which are in $\mathbb{B}_t$ since, if $j\in V_s$ is the label of the row in question, the row vector can be formed by a single sweep of a path minimal to vertex j through to the next path maximal to j. Thus the condition on elements of Q_t becomes the l_∞ condition required.

(ii) Immediate from (i).

(iii) No element of Q is infinitely divisible, nor is it torsion.

Thus $Q\equiv 0$ in many simple cases, e.g. in the stationary case when there are no eigen-vectors with value ±1 or in general when there are no infinitesimals in $K_0(I)$. Indeed it is quite hard to find examples where Q is non-zero, but they do arise. See example (d) later.

THEOREM 9: *Suppose that I is an ordered Bratteli diagram with a finite number of maximal and minimal paths, then there is a finite $\nu\ge 0$ such that, as groups,*

$$K_0(I)/Q\oplus\mathbb{Z}^\nu\equiv K^0(\Pi,S).$$

Proof: This splits into several sections labelled as follows:

(A) Sets which span $C(\Pi\colon\mathbb{Z})$: (I) Indicators of basic clopen sets and products of certain pairs of these are sufficient to span $C(\Pi\colon\mathbb{Z})$. (II) The space, B, spanned by the basic indicators and simple relations between the pair products modulo B. (III) The linear span of these simple relations generate all there are.

(B) Direct sum decompositions and coboundaries: (I) B is complemented in $C(\Pi\colon \mathbb{Z})$ which splits into a direct sum of B and C, both S-invariant spaces. (II) $B/\operatorname{cbdy}_S(B) \equiv K_0(I)/Q$. (III) $C/\operatorname{cbdy}_S(C) \equiv \mathbb{Z}^\nu$.

(AI) Recall the functions $F_p^{(t)}\colon \Pi \to \{0,1\}$ which indicate those path sequences (p_n) for which p_0 agrees with p to the tth. level. The algebraic span of these and all their shifts is $C(\Pi\colon \mathbb{Z})$. The number of multiplicative combinations needed to span $C(\Pi\colon \mathbb{Z})$ can be reduced significantly by the following considerations:

The $F_p^{(n)}$ themselves are not linearly independent. In fact each $F_p^{(n)}$ can be written uniquely as a finite sum of $F_{p'}^{(n+1)}$ functions of disjoint support, namely those indexed by p' which are paths agreeing with p to the nth. level and disinct to the $n+1$st. level.

Moreover, given two paths p,q the product $F_p^{(n)}F_q^{(m)}$, $n \leq m$, equals the zero function except in the case that p agrees with q up to the nth. level, in which case the product equals $F_q^{(m)}$. I.e. products of basic zero-coordinate functions span nothing more than the basic functions would span.

Consider two paths p,q with p not maximal to level n. In this case, given another path p', p' agrees with p to the nth. level if and only if Vp' agrees with Vp to the nth. level. Thus $F_p^{(n)} = F_{Vp}^{(n)} \circ S$ and the product $F_p^{(n)}(F_q^{(m)} \circ S)$ comes under the argument of the last paragraph and so reduces to a single $F_{p'}^{(n')} \circ S$ or zero.

On the other hand, if p is maximal to the nth. level, then either p agrees with an infinite maximal path to the nth. level or there is an $n' > n$ such that every path which agrees with p to the nth. level is not maximal to the n'th. level. In the latter case, $F_p^{(n)}$ is a sum of expressions of the form $F_{p'}^{(n')}$, none of the p' maximal to the n'th. level, and the argument of the previous paragraph applies here to the same effect.

The case that remains is the one where p is, in effect, an infinite maximal path. The argument above applies equally to q with respect to minimality and so the only case unaccounted is the one where p is maximal and q minimal. Assume that this is so and assume further that p is not related to q by $\sim$. Consider the product $F_p^{(n)}(F_q^{(m)} \circ S)$:

First, recall the assumption that pairs of maximal paths have no edge in common. Thus, if p is maximal $F_p^{(n)}$ can be expressed as a sum of functions of the form

$F_{p'}^{(n+1)}$, where precisely one of the p' is maximal and equal to p therefore. Similarly for the $F_q^{(n)}$ with respect to minimality. Thus, for $n' > n, m$, $F_p^{(n)}(F_q^{(m)} \circ S)$ is a sum of products like $F_{p'}^{(n')}(F_{q'}^{(n')} \circ S)$, where p' is not maximal or q' is not minimal, plus $F_p^{(n')}(F_q^{(n')} \circ S)$. All these first summands are single $F_{p'}^{(n')}$ or identically zero by the arguments of past paragraphs and so $F_p^{(n')}(F_q^{(n')} \circ S)$ differs from $F_p^{(n)}(F_q^{(m)} \circ S)$ by a sum of functions of the form $F_{p'}^{(n')}$.

Now pick n' sufficiently large that $F_p^{(n')}(r) = 1$ implies $F_q^{(n')}(Sr) = 0$. This can be done by the fact that p and q are unrelated by $\sim$. Thus $F_p^{(n')}(F_q^{(n')} \circ S) = 0$ and $F_p^{(n)}(F_q^{(m)} \circ S)$ is a sum of functions of the form $F_{p'}^{(n')}$.

The case $p \sim q$ will be left necessarily.

This reduction argument may be applied inductively to longer products to show that nothing more than $F_p^{(n)} \circ S^j$: $p \in P, n \in \mathbb{N}, j \in \mathbb{Z}$ and $(F_p^{(n)} \circ S^j)(F_q^{(n)} \circ S^{j+1})$: $p \sim q, n \in \mathbb{N}, j \in \mathbb{Z}$ are needed to span $C(\Pi\colon \mathbb{Z})$ linearly.

(AII) Let $B = \operatorname{span}_{\mathbb{Z}}\{F_p^{(n)} \circ S^j\colon p \in P, n \in \mathbb{N}, j \in \mathbb{Z}\}$ and $DD = \operatorname{span}_{\mathbb{Z}}\{(F_p^{(n)} \circ S^j)(F_q^{(n)} \circ S^{j+1})\colon p \sim q, n \in \mathbb{N}, j \in \mathbb{Z}\}$. The arguments above show that $C(\Pi\colon \mathbb{Z}) = B + DD$ therefore.

The argument now reduces this spanning set further: Consider the linear interdependences of the $(F_p^{(n)} \circ S^j)(F_q^{(m)} \circ S^{j+1})\colon p \sim q$ modulo B.

First recall the work of previous paragraphs relating to the simplification of products: This shows that, for all n, m,

$$F_p^{(1)}(F_q^{(1)} \circ S) = F_p^{(n)}(F_q^{(m)} \circ S) + \sum F_{p'}^{(n')}$$

where the latter summand is a finite sum over p' none of which is maximal to the n'th. level. This expression will be used constantly in what follows.

In particular, $C(\Pi\colon \mathbb{Z}) = B + D$ where

$$D = \operatorname{span}_{\mathbb{Z}}\{(F_p^{(1)} \circ S^j)(F_q^{(1)} \circ S^{j+1})\colon p \sim q, j \in \mathbb{Z}\}.$$

Another set of linear relations is obtained: for each m there is an n so that for fixed $p \in M$

$$\sum_{q\colon p \sim q} F_p^{(n)}(F_q^{(m)} \circ S) = F_p^{(n)} \in B$$

and similarly for fixed $q \in N$

$$\sum_{p\colon p \sim q} F_p^{(m)}(F_q^{(n)} \circ S) = F_q^{(n)} \circ S \in B.$$

To see the first equality, for example, it is sufficient to have n large enough so that if p is maximal and $F_p^{(n)}(r) = 1$, then $F_q^{(m)}(Sr) = 1$ for some q: $p \sim q$. Similarly for the second equality.

Using the equivalences above, this becomes

$$(*) \qquad \sum_{q:\, p\sim q} F_p^{(1)}(F_q^{(1)} \circ S) = 0 \bmod B$$

for fixed $p \in M$ and

$$(*) \qquad \sum_{p:\, p\sim q} F_p^{(1)}(F_q^{(1)} \circ S) = 0 \bmod B$$

for fixed $q \in N$. Other linear relations are obtained by applying some power of S to these relations.

(AIII) Now suppose that $\sum_{p,q,j} a(p,q,j)(F_p^{(1)} \circ S^j)(F_q^{(1)} \circ S^{j+1}) = 0 \bmod B$ where all but finitely many $a(p,q,j)$ are zero. The aim is to prove that this equality can be described as a finite linear combination of the relations $(*)$:

Write this out in full:

$$\sum a(p,q,j)(F_p^{(1)} \circ S^j)(F_q^{(1)} \circ S^{j+1}) = \sum b(p,n,j)(F_p^{(n)} \circ S^j) \in B$$

where it is understood from now on that such sums run over all integer variables which are explicit arguments of the coefficients and all but finitely many of the summands are zero.

As in previous arguments, the $F_p^{(n)}$ on the right-hand side may be decomposed into $F_{p'}^{(n')}$ where either p' is not maximal to the n'th. level, or p' is a maximal path. This can be represented as a splitting of the sum on the right-hand side itself into Σ^1 and Σ^2 respectively.

Σ^2, involving maximal paths, can be split further using the relations above, as Σ^{2A} which is a sum over $p" \sim q"$ of the form $\sum a'(p",q",j)(F_{p"}^{(s)} \circ S^j)(F_{q"}^{(s)} \circ S^{j+1})$ for s chosen large enough, and Σ^{2B} which is a sum involving $(p",n")$ such that $p"$ is not maximal to the $n"$th. level. Note that the a' are formed from direct linear combinations of the relations therefore.

On the left-hand side: Suppose that $\max\{|j|\colon\ \exists\ (p,q)\colon a(p,q,j) \neq 0\} = J$. Recall the definition of $x(p,q)$, points in Π, from before. Find t large enough so that, if $p \in M$ and $|j|, |j'| \leq J$, $F_p^{(t)}(S^{j+j'}x(p_0,q_0)) = 1$ implies that $j+j' = 0$ and

$p_0 = p$ to level t, and so that, if $q \in N$ and $|j|, |j'| \le J$, $F_q^{(t)}(S^{j+j'+1}x(p_0, q_0)) = 1$ implies that $j + j' = 0$ and $q_0 = q$ to level t.

Here the fact that there are a finite number of maximal paths and minimal paths is used.

Using the connection between $F_p^{(1)}(F_q^{(1)} \circ S)$ and $F_p^{(t)}(F_q^{(t)} \circ S)$ noted before, the left-hand side can be re-written as two sums: Σ^3 involving $F_{p'}^{(n')}$ with p' not maximal to the n'th level; and $\Sigma^4 = \sum a(p,q,j)(F_p^{(t)} \circ S^j)(F_q^{(t)} \circ S^{j+1})$ where it is essential to note that the $a(p,q,j)$ are the same coefficients as before.

Without loss of generality, $t = s$ and this common value is greater than the n' and n" involved above.

The equality above can be rearranged therefore as

$$\Sigma^4 - \Sigma^{2A} = \Sigma^1 + \Sigma^{2B} - \Sigma^3.$$

In full

$$\sum (a'(p,q,j) - a(p,q,j))(F_p^{(t)} \circ S^j)(F_q^{(t)} \circ S^{j+1}) = \sum b'(p',j)(F_{p'}^{(t)} \circ S^j)$$

where the sum on the right involves only p' which are not maximal to the tth. level, and the sum on the left involves only $p \sim q$.

Evaluate both sides of this at the point $S^{j'}x(p_0, q_0)$ and use the construction of t to show that $a'(p_0, q_0, -j') - a(p_0, q_0, -j') = 0$ for all $p_0 \in M$, $q_0 \in N$ and $|j'| \le J$.

Recall that the a' are formed from linear combinations of the relations $(*)$ and so the present aim of this argument is achieved: The relations $(*)$ between the $F_p^{(1)} \circ S^j F_q^{(1)} \circ S^{j+1}$ modulo B above span linearly all the relationships possible.

(BI) Let

$$D = \operatorname{span}_{\mathbb{Z}}\{(F_p^{(1)} \circ S^j)(F_q^{(1)} \circ S^{j+1}) \colon p \sim q, j \in \mathbb{Z}\}.$$

Define

$$D_j = \operatorname{span}_{\mathbb{Z}}\{(F_p^{(1)} \circ S^j)(F_q^{(1)} \circ S^{j+1}) \colon p \sim q\}$$

so that $D = \sum D_j$ and $D \cap B = \sum (D_j \cap B)$.

Consider the graph G whose vertex set is $M \cup N$ and in which p is connected to q if and only if $p \in M, q \in N$ and $p \sim q$. Note that by the natural correspondence of pairs (p,q) with edges of G, the columns of the edge-vertex matrix, $A(G)$, are precisely the relations $(*)$ (for fixed j) involved in (AII). Thus $D_j \cap B$ sits in D_j exactly as $\mathrm{Im}(A(G))$ sits in $\mathbb{Z}^E$.

G is bipartite and so Lemma 6 shows that $D_0 = (D_0 \cap B) \oplus C_0$ where C_0 is the span of some unit vectors in D_0 which correspond to some minimal set of edges, E', whose removal makes G a forest. Let $\{H_1, \ldots, H_\nu\}$ be a basis set for C_0 where each H_i is a single $F_p^{(1)} F_q^{(1)} \circ S$, $(p,q) \in E'$. Similarly, let $C_j = \mathrm{span}_{\mathbb{Z}}\{H_1 \circ S^j, \ldots, H_\nu \circ S^j\}$ and note $D_j = (D_j \cap B) \oplus C_j$.

The work in (AIII) shows that $D/(D \cap B) = \oplus D_j/(D_j \cap B)$ or, in other words, the C_j are linearly independent. Let $C = \oplus_j C_j$ so that $\bigcup\{H_1 \circ S^j, \ldots, H_\nu \circ S^j\}$ is an algebraic basis of C. Thus $C \cap B = 0$ and it is clear that $C + B = C(\Pi\colon \mathbb{Z})$. So $C(\Pi\colon \mathbb{Z}) = B \oplus C$ is a decomposition into S-invariant subspaces.

(BII) Now the aim is to determine the effect of quotienting out the coboundaries from $C(\Pi\colon \mathbb{Z})$:

First note that the quotient can be split between B and C above with the help of S-invariance, i.e. $K_0(\Pi, S) = B/\ \mathrm{cbdy}_S(B) \oplus C/\ \mathrm{cbdy}_S(C)$.

Write $F \simeq F'$ if there is $G \in C(\Pi\colon \mathbb{Z})$ such that $F - F' = G - G \circ S$. The last remark implies that if both F and F' come from B, then G can be picked from B as well.

Note that $F_p^{(n)} \simeq F_q^{(n)}$ whenever p and q pass through the same vertex at level n. To see this, observe that there is a $j \in \mathbb{Z}$ such that a path, p', agrees with p to level n if and only if $S^j p'$ agrees with q to level n. Thus $F_p^{(n)} = F_q^{(n)} \circ S^j$ and $G = \sum_{i=1}^{j} F_q^{(n)} \circ S^j$ if $j \geq 0$ and $G = -\sum_{i=j+1}^{0} F_q^{(n)} \circ S^j$ if $j < 0$ will do.

Also note that $F \simeq F \circ S$.

Recall the unordered Bratteli diagram and the matrices $J^{(n,t)}$ which count the number of paths from points in level n to points in level t. Consider the vectors e_v, v a vertex in level n, which form the natural unit basis for $\mathbb{Z}^{V_n}$, the domain of $J^{(n,t)}$ for all $n < t$. Recall also the notation $p(n)$: the vertex at level n through which path p passes.

To show a (1-1) linear correspondence between $K_0(I)/Q$ and $B/\ \mathrm{cbdy}_S(B)$, it is sufficient to show that $\sum a_i(F_{p_i}^{(n_i)} \circ S^{k_i}) \simeq 0$ iff there is a $t > n_i$ such that $\sum a_i J^{(n_i,t)} e_{p_i(n_i)} \in Q$, for then the map $F_p^{(n)} \mapsto [(n, e_{p(n)})]$ extends linearly $C(\Pi\colon \mathbb{Z}) \to K_0(I)$ and factors through the quotients.

Without loss of generality, $n_i = s$ for all i; both expressions above transform in the same way when summands are split up to a common higher level. Also, all mention of S can be removed easily by coboundaries. Further, the arguments above make $F_{p_i}^{(s)}$ here dependent (modulo coboundaries) only on the vertex at level s through which p_i passes. So suppose that, for each $v \in V_s$, p_v is an infinite path which passes through v at level s, which is maximal up to that level and arbitrary at higher levels. Then the sum of interest, transformed now to $\sum a_i F_{p_i}^{(s)}$, can be regrouped into a $\simeq$-equivalent sum $\sum_{v \in V_s} a'_v F_v^{(s)}$, where $F_v^{(s)} = F_{p_v}^{(s)}$.

It is sufficient, therefore, to show that $\sum_{v \in V_s} a_v F_v^{(s)} \simeq 0$ if and only if $\sum a_v e_v \in Q_s$.

A well-known result of Gottschalk and Hedlund ([GH]) makes the correspondence immediate: Their result shows that $F \simeq 0$ if and only if there is a constant K such that $|\sum_{k=0}^{L} F(S^k(q_n))| \leq K$ uniformly for all $L \geq 0$ and all $(q_n) \in \Pi$.

Putting $F = \sum_{v \in V_s} a_v F_v^{(s)}$ in the condition above is equivalent to the definition of Q_s almost immediately and gives the correspondence required:

$K_0(I)/Q \equiv B/\operatorname{cbdy}_S(B)$.

(BIII) The coboundary behaviour of C is easier to compute directly from the linear relations determined before: Removal, by coboundaries, of the mention of S ensures that it is sufficient to consider expressions of the form $\sum a_i H_i = G - G \circ S$ where $G \in C$.

When an equality of the form $\sum a_i H_i = G - G \circ S$ where $G \in C$ is written out as a sum of $(F_p^{(1)} \circ S^j)(F_q^{(1)} \circ S^{j+1})$ functions, the left-hand side involves only $j = 0$, but the right-hand side must involve other $j \neq 0$ and so, by the linear independence observed in (BI), both sides are zero. Thus the equivalence classes $\{[H_i]_{\operatorname{cbdy}_S(C)} : i = 1, \ldots, \nu\}$ form a basis for $C/\operatorname{cbdy}_S(C)$ as required.

So the theorem is complete.

Corollary 10 of Proof: *$\nu = e - v + c$ where e is the number of $\sim$ equivalent pairs of maximal paths and minimal paths, v is the number of maximal or minimal paths and c is the number of components in the graph G whose vertices are maximal or minimal paths connected by the $\sim$ relation.*

Note that all the decompositions, quotients etc. are performed by positive operations and some of the order structure is preserved. However, there seems to be no easy general way to compute the positive cone and unit in this representation of $K^0(\Pi, S)$. All that can be deduced from the construction of B and C above is that the direct sum positive cone of $K_0(I)/Q \oplus \mathbb{Z}^\nu$ is contained in the positive cone of $K^0(\Pi, S)$ under the isomorphism above. For example, if $\nu = 0$ then $K_0(I)/Q$ is order isomorphic to $K^0(\Pi, S)$.

Application to substitution minimal systems

This final section gives the motivating application of the general results of the last section: The computation of the K^0 group without order for a given primitive substitution minimal dynamical system. Using this and Theorem 3, these dynamical systems may be distinguished therefore at the level of strong orbit equivalence.

The work of the previous section applies using the following lemma:

LEMMA 11: *Suppose that σ is a primitive substitution with alphabet A and $\omega = \omega(a.i)$ is a stationary sequence which generates the substitution minimal system $(X(\omega), S)$ as before. Suppose that I is the improper Bratteli diagram for σ and that Π is the path-sequence space constructed from I before. Then there is $p \sim q$ and ψ such that ψ: $(\Pi, S, x(p,q)) \to (X(\omega), S, \omega)$ is a pointed topological conjugacy.*

Proof: Recall the construction of the improper Bratteli diagram for the substitution σ defined on an alphabet A. Recall also the map, π, defined on the path space, P, which records the letter which corresponds to the vertex at the first level through which the path passes.

The aim of the proof is to show that the map ψ from Π to $A^{\mathbb{Z}}$ defined by ψ: $(p_n) \to (\pi(p_n))$ is homeomorphic onto $X(\omega)$. If this is true, then this map clearly respects the shift dynamics and conjugacy follows.

Some facts from before: Recall that this diagram is minimal in the sense that, given a pair of paths p, q and a $t \geq 1$, some well-defined $V^i p$ will agree with q to level t. π assigns distinct values to maximal paths and distinct values to minimal paths. Also $\omega(a.i)$ can be retrieved as $\cdots \pi(V^{-1}p^*)\pi(p^*).\pi(q^*)\pi(Vq^*)\cdots$, for some p^* maximal and some q^* minimal. $\omega^+ = .\pi(q^*)\pi(Vq^*)\cdots$.

(i) ψ maps to $X(\omega)$: Suppose that $(p_n) \in \Pi$ and let $\alpha = \psi((p_n)) = (\pi(p_n)\colon n \in \mathbb{Z})$. To show that $\alpha \in X(\omega)$ it is enough to show that every finite subword of α is a subword of ω and it is sufficient to consider the finite words $\alpha[-N, N]$ for N fixed large. By the minimality of (Π, S) there is an $m \geq 0$ such that $V^m q^*$ is sufficiently close to p_{-N} that $\pi(V^{m+i}q^*) = \pi(p_{-N+i})$ for all $0 \leq i \leq 2N$. Therefore, $\alpha[-N, N] = \omega^+[m, m+2N]$ as required.

(ii) ψ is continuous: This is clear from the continuity of π on the path space.

(iii) An inverse to ψ: Suppose that (p_n) and α are as above and that $k \geq 1$. Suppose that $M_k = \max\{|\sigma^k(b)|\colon b \in A\}$ and set $L_k = L(k, M_k)$ from Corollary

2. This implies that wherever $\alpha[-L_k, L_k] = \omega^+[t - L_k, t + L_k]$ appears in ω^+, the k beat is determined to radius M_k around position t. This is sufficient radius to determine uniquely the kth. order substitution word, β_k, which sits around the 0 coordinate in α. β_k inherits a natural zeroing from its position around the zero in α. There a unique symbol b_k so that $\beta_k = \sigma^k(b_k)$ ignoring the zeroing.

Thus, knowing α determines a sequence of symbols b_k which may be interpreted as vertices in V_{k+1}. The symbol $b_0 = \alpha_0$, by convention, a vertex in V_1. The edges which are constructed to join these vertices can be found in the improper diagram and are determined by the relative positions that the zeroed words β_k nest in each other. So, from α, a path, $\rho(\alpha)$, in P can be found. Also ρ is continuous since the construction of initial segments of $\rho(\alpha)$ depends only on a fixed block of coordinates from α.

The construction above is unique and well-defined at each stage by Corollary 2.

Note also that if $V^n\rho(\alpha)$ is defined, then it equals $\rho(S^n\alpha)$; the β_k pattern is simply shifted n places.

In constructing $\rho(\alpha)$ two things can happen:

(a) The nested zeroed words β_k expand unboundedly both left and right so that $\lim \beta_k = \alpha$. In this case it is clear that $\rho(\alpha)$ is a path whose forward and reverse iterates by the Vershik map are all defined and further $\pi(V^n\rho(\alpha)) = \alpha_n$.

(b) The nesting is bounded on one side and $\lim \beta_k$ is a semi-infinite zeroed subword of α. Say that the nesting is bounded to the left and the extreme point is the symbol at the $-m$th coordinate of α, $m \geq 0$. This implies that all edges in $\rho(\alpha)$ are minimal beyond a certain level: the level beyond which each β_k nests at the extreme left of β_{k+1}. Thus all edges in $q = \rho(S^{-m}\alpha)$ are minimal and so q is a minimal path in the improper diagram.

The nesting pattern for $S^{-m-1}\alpha$ is also bounded, but on the right, and all the edges in $p = \rho(S^{-m-1}\alpha)$ are maximal and p is a maximal path, therefore.

Since $S^{-m}\alpha \in X(\omega)$, $\alpha[-m - L, -m + L] = \omega[t - L, t + L]$ is a subword of ω for any choice of L and many choices of $t > L$. In particular let $L = L_k$ as before, which is large enough to determine uniquely the level k substitution words either side of the tth. coordinate in ω whatever t is picked. Let r be the path $\rho(S^t\omega)$ and then note that r agrees with q and $V^{-1}r$ agrees with p both to level k by construction. k was arbitrary and so $p \sim q$.

Therefore, in this case, the sequence of paths $p_n = \rho(S^n\alpha)$ is also a member of Π with a $\sim$ break between the $-m-1$ and $-m$ coordinate where the Vershik map is not defined.

The upshot of this is that the map ϕ from $X(\omega)$ to path sequences, defined $\phi((\alpha)_n\colon n\in\mathbb{Z})=(\rho(S^n\alpha)\colon n\in\mathbb{Z})$, maps to Π.

It is not hard to see now that ϕ and ψ are inverses and continuous maps, hence the homeomorphism required above.

Incidentally, since ω itself comes under case (b) above with the break between -1 and 0, this shows that $p^*\sim q^*$, a fact that was not assumed in the arguments above.

COROLLARY 12 OF PROOF: *There is a bijection, κ, between the set $\{(p,q)\colon p\sim q\}$ and the set of recurrent pairs, bc, to be found in ω (i.e. bc such that $\omega(b.c)$ is well-defined and in $X(\omega)$). This bijection is well-defined by $\phi(\omega(b.c))=x(p,q)$ if and only if $\kappa(p,q)=bc$ and, in fact, $\kappa(p,q)=\pi(p)\pi(q)$.*

Proof: This comes from considering case (b) in the proof above. All instances of this case correspond equally to recurrent pairs of letters to be found in ω and $\sim$-related maximal and minimal paths in the diagram.

SIMPLIFYING ν AND Q FOR SUBSTITUTION SYSTEMS. Define the following graph $G'=(V',E')$: $V'=\Lambda\cup\Lambda'$ is two copies of Λ and $b\in\Lambda$ is connected to $c'\in\Lambda'$ if and only if bc is recurrent and appears in ω. Recall that $\lambda=|\Lambda|$.

Recall the graph, G, defined from maximal and minimal paths in the improper Bratteli diagram (Corollary 10). Corollary 12 above shows that $G'=G\cup\{\text{isolated points}\}$ and so $A(G)$ differs from $A(G')$ by a few zero columns. Thus $\mathrm{Im}(A(G'))=\mathrm{Im}(A(G))$ etc., and all the constructions of the past section proceed with reference to G' instead of G.

In the case of substitutions the description of sweep vectors may be simplified as well: given the substitution sequence $\omega\in\Lambda^{\mathbb{Z}}$, let W be the set of finite subwords of ω. For each $\alpha\in W$ define a vector $\beta\in\mathbb{Z}^{\Lambda}$ with coordinates β_b equal to the number of occurrences of the letter a in α. The collection of such vectors coincides with the subset of positive vectors in $\mathbb{B}_t$ for any t.

The presence of non-zero vectors in Q reflects therefore a strict rational balance between the numbers of occurrences of individual letters in the substitution sequence. See example (d) below.

These simplifications are of particular use when going directly from the substitution to the dimension group without reference to Bratteli diagrams. The following is a rephrasing of results 9 to 12.

THEOREM 13: *If σ is a primitive substitution scheme with all other notation as before, then $K^0(X(\omega),S) = K_0(I)/Q \oplus \mathbb{Z}^\nu$ where $\nu = e' - 2\lambda + c'$, e' being the number of edges in G' and c' the number of components.*

Examples

The Theorems above have proved very useful in simplifying the computation of dimension groups of given substitution minimal systems. Whereas the proper Bratteli diagram is often difficult to compute and the connection matrix is sometimes of high dimension, the improper diagram can be written down immediately and is often of small dimension. This is best illustrated by examples:

(a) The Morse-Thue substitution minimal system: The improper diagram connection matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

has no unit eigen-value and so $Q \equiv 0$. $K_0(I) = \mathbb{Z}[1/2]$. The matrix $A(G) = A(G')$ in this case is

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and from this can be computed $\nu = 1$. Thus $K^0 \equiv \mathbb{Z}[1/2] \oplus \mathbb{Z}$. This may be confirmed long-hand using the connection matrix for the proper diagram noted before.

(b) The substitution $0 \to 001$ and $1 \to 110$ is primitive etc. forming a sequence starting 0.0 and using σ^2. The connection matrix for the Improper Bratteli Diagram has eigen-value 1 corresponding to an infinitesimal $(1,(1,-1))$ in the K_0 group and allows the possibility that Q can be non-trivial. However, an induction argument shows that the number of 0s in the word $\omega^+[0, 2.3^n]$ exceeds the number of 1s by $n+2$ and so $Q \equiv 0$ by examining the vectors in $\mathbb{B}_1$ defined by such words (there is no need to examine higher $\mathbb{B}_t$ by stationarity). $K_0(I) \equiv \mathbb{Z}[1/3] \oplus \mathbb{Z}$. Also G' is the same as for the Morse sequence, so $K^0 \equiv \mathbb{Z}[1/3] \oplus \mathbb{Z}^2$.

(c) The K_0 for the proper Bratteli diagram from the substitution $0 \to 011110$, $1 \to 100001$, is $\mathbb{Z}[1/2] \oplus \mathbb{Z}[1/6] \oplus \mathbb{Z}$.

(d) The substitution $0 \to 021, 1 \to 120, 2 \to 21202$, with the recurrent pair 2.0, is a non-trivial example of a system in which $Q \neq 0$: Here the improper diagram connection matrix is

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 3 \end{pmatrix}$$

and $K_0(I) = \mathbb{Z}[1/2] \oplus \mathbb{Z}$, the second summand being generated by the eigen vector $(1,1,-1)$ with eigen-value 1. Indeed, the periodicity of the symbol 2 in the sequence ensures that Q is generated by $(1,(1,1,-1))$. The graph for G' has two components each of which is a two-edge path. Thus $\nu = 0$ and so K^0 for this case equals $\mathbb{Z}[1/2]$, implying that this system is strongly Kakutani orbit equivalent to the 2-adic odometer.

Stationary ordered Bratteli diagrams and substitution systems

This final section achieves a complete characterisation of the strong orbit equivalence classes of substitution minimal sets in terms of certain dimension groups with unit. It also characterises the conjugacy classes of substitution minimal systems in terms of certain stationary ordered Bratteli diagrams.

Unlike past sections, this one will need to distinguish carefully between ordered Bratteli diagrams and Bratteli diagrams without the order. There is a notion of telescoping and equivalence of diagrams (see [HPS, GPS, S]) within each of these two categories. The reader will be assumed to be familiar with these constructions in this section. The main result used is:

THEOREM 14 (from [HPS, S]): *Suppose that $(X,S,x),(Y,T,y)$ are Cantor minimal systems with distinguished points. Suppose further that I,J are the respective proper ordered Bratteli diagrams, and $K_0(I),K_0(J)$ the respective dimension groups with unit (the order structure is now important).*

(a) *(X,S) is strong orbit equivalent to (Y,T) if and only if $K_0(I) \equiv K_0(J)$ as dimension groups with unit.*

(b) *(X,S,x) is pointed conjugate to (Y,T,y) if and only if I is order-equivalent to J.*

The natural construction in [LV] of an improper diagram from a given substitution can be reversed. Given a minimal stationary ordered Bratteli diagram, a substitution can be read from the repeating graphical unit in a natural way: Each level is in natural 1-1 correspondence with a set of letters and the substitution word for a given letter is determined by the source of the edges, taken in order, whose common range corresponds to the given letter.

Ideally this substitution is primitive and the diagram is proper, allowing Theorem 11 to show that the Vershik map is conjugate to the corresponding substitution minimal system by a natural homeomorphism. The results of this section show how far this ideal applies in general.

Minimality of the diagram forces the resulting substitution to satisfy many of the conditions required for primitivity. It helps further when the diagram is proper. But there remain problems in general when checking the primitivity of the substitution derived from the diagram, in particular when hoping to exclude periodicity in the substitution sequence. This complication will be met later.

First, another problem can be dealt with immediately: The improper diagram for a substitution always has a single edge between the vertex at the 0th. level and each vertex in the 1st. level. It is useful to be able to reduce the general stationary Bratteli diagram with multiple edges at this level to one with single edges.

LEMMA 15: *Suppose that I is a minimal stationary ordered Bratteli diagram, then it is order-equivalent to $J = (V_n, E_n)$, a minimal stationary ordered Bratteli diagram in which each vertex of V_1 is connected to V_0 by a single edge. If I is proper, then J is proper also.*

Proof: Suppose that C is the connection matrix of the repeating unit of I and that $v = (v_1, v_2, \ldots, v_m)^T$ is the column vector which represents the connections of level 0 to level 1. Let $M = \sum v_i \ (\geq m)$.

Let A be the $m \times M$ matrix whose jth column $(1 \leq j \leq M)$ is the unit vector $(0, 0, \ldots, 1, ..0)^T$ with 1 in the kth. place for all k such that $\sum_{i<k} v_i < j \leq \sum_{i \leq k} v_i$. There are many choices of B, an $M \times m$ matrix, so that $AB = C$. Replacing C with some high power itself if necessary, one can choose B with all its entries strictly positive. Thus $D = BA$ is an $M \times M$ matrix whose entries are all strictly positive. Also $A(1, 1, \ldots, 1)^T = v$.

Thus the stationary minimal unordered diagram, J, with connections D between levels ≥ 1 (each with M vertices) and $(1, 1, \ldots, 1)^T$ between level 0 and level 1, is equivalent to the unordered diagram which lies under I since they are both a telescoping of an intermediate diagram with periodic connection matrices: $(1, 1, .., 1)^T, A, B, A, B \ldots$.

In this intermediate diagram, the order on the edges between levels with connection matrix A is unique, and an order for edges with connection matrix B can be assigned so that the telescoping of BA to C also reproduces the original order of I provided that we have been careful in the choice of B. The reader should try a simple example to be convinced that with enough edges available from B such a choice can be made in general. This defines an order for the telescoping of AB to D and hence for J. So we are done.

Equivalence preserves the minimal paths and maximal paths in a natural way and so preserves the properness.

This latter argument is in fact a special case of symbol splitting: See [HPS] and [S] for the introduction of this notion.

A convenient way of ensuring that a minimal stationary ordered Bratteli diagram is proper is to provide all the maximal edges with range in V_2 with a common source in V_1 and likewise for the minimal edges. Thus every substitution word in the derived substitution starts with the same letter and ends with the same letter, e.g. $0 \to 0011, 1 \to 0101$. Given enough edges between points in consecutive levels, there is great freedom to order the edges in this way while ensuring that the corresponding substitution is primitive. This idea gives almost immediately the proof of the following Theorem:

THEOREM 16: *Suppose that C is an irreducible square ($\lambda \times \lambda$, $\lambda \geq 2$) matrix with strictly positive integer entries, $K = \lim_C \mathbb{Z}^\lambda$ with a given order unit ($= [(1, v)]$, all coordinates of v strictly positive, without loss of generality). Then there is a minimal primitive substitution system, (X, S), on M letters, where M is the sum of the coordinates of v, such that $K^0(X, S) \equiv K$ as dimension groups with unit.*

Proof: Construct a minimal unordered Bratteli diagram with C as the connection matrix and with multiple edges between level 0 and level 1 according to the coordinates of v. The dimension group with unit of this diagram is isomorphic to K.

Use the argument of Lemma 15 (ignoring the order structure) to make an equivalent unordered diagram with single edges between level 0 and level 1. There are M of these edges by construction and M vertices at each level. Equivalence of the diagrams ensures that the dimension group with unit that corresponds to this diagram is isomorphic to K and so, without loss of generality, it may be assumed that there are single edges between level 1 and 0.

Now, replacing C with some high enough power if necessary, assign an order on the edges to ensure that the ordered diagram is proper and the corresponding substitution is primitive. As noted above, the substitution system derived from this will be conjugate to the Vershik map which has K^0 equal, as dimension group with unit, to K_0 of the diagram as required.

Thus, by Theorem 14a, the strong orbit equivalence classes of substitution minimal systems are characterised by dimension groups with unit which are simple stationary limits of the form $\lim_C \mathbb{Z}^n$ with given unit.

Theorem 16 leaves much of the order structure of a stationary ordered Bratteli diagram unexploited. In fact, a complete dynamical characterisation of proper minimal stationary ordered diagrams is possible. This shows that not only can any K_0 group of such a diagram be described as a K^0 group of some substitution minimal system, but that every stationary proper order on the diagram has its Vershik map either equicontinuous or conjugate to a substitution minimal system.

THEOREM 17: *Suppose that I is a minimal stationary ordered proper Bratteli diagram (and non-trivial in the sense that the Markov Compactuum is a Cantor Set) and that (X, S) is the well-defined Vershik map on the Markov Compactuum. Then (X, S) is either*

(a) *Kakutani equivalent to a d-adic system for some $d \geq 2$, or*

(b) *conjugate to a substitution minimal system.*

Hence, under the assumtions above: (X, S) is equicontinuous iff (X, S) is Kakutani equivalent to a d-adic system; (X, S) is expansive iff (X, S) is conjugate to a substitution minimal system.

Proof: This uses frequently and without mention the result of Theorem 14b: that proper ordered diagrams, equivalent by telescoping, have conjugate dynamics.

Assume, without loss of generality, that there are at least 2 vertices at the first level. Otherwise, the system is clearly Kakutani equivalent to a d-adic system where d is the number of edges between level 1 and 2 say.

Replace I with the J of Lemma 15, which is a proper minimal diagram, therefore, with single edges between levels 0 and 1. The unique maximal path passes through a sequence of vertices as it progresses through the levels of the diagram. The sequence of corresponding letters is a one-sided periodic sequence and the diagram can be telescoped periodically so that, without destroying stationarity nor the single edge from level 0 to level 1 property, the maximal path goes through a sequence of vertices corresponding to the same letter, a say. Similarly for the minimal path and letter i (which may be the same as a) with more periodic telescoping if necessary.

Further periodic telescoping ensures that, without loss of generality, every minimal edge in E_n has as its source the vertex in V_n which corresponds to the symbol i. Similarly for a with respect to maximal edges. A substitution σ can be read from the diagram as mentioned before. The construction above implies that $\sigma(b) = i \cdots a$ for all symbols b and minimality of the diagram shows that ai is a recurrent pair for the substitution.

As noted before, the proof would be finished if this substitution were primitive, but this need not happen. The main difficulty is the possibility that the sequence $\lim \sigma^n(a.i)$ is periodic. The proof proceeds by showing that the periodic case gives option (a) in the statement of the theorem.

Suppose that the sequence $\omega = \lim \sigma^n(a.i)$ is periodic and that β is the shortest repeating unit: $\omega = \cdots \beta\beta.\beta\beta \cdots$. Assume that n is chosen large enough so that for each symbol, b, $\sigma^n(b)$, a subword of ω, contains sufficiently many mentions of β. Suppose that $\sigma^n(i) = \beta\beta \cdots \beta\beta'$ where β' is a proper initial segment of β. β' terminates with a and so $\sigma^{2n}(i) = \cdots \sigma^n(a) = \cdots \beta\beta\beta$. If the length of $\sigma^{2n}(i)$ were not a multiple of the length of β, then ω would equal itself offset by a shift smaller than the length of β. This would contradict the minimality of β. A similar argument applies to sufficiently high powers of σ applied to each of the symbols. So, by telescoping the diagram periodically sufficiently many times more, it can be assumed that each $\sigma(b)$ has the form of a periodic finite word $\beta \cdots \beta$.

This allows a further modification of the diagram: For each $n \geq 1$, interpolate a level, W_n, between V_n and V_{n+1}, consisting of a single point. For each symbol b, connect the vertex in V_{n+1} which corresponds to b to the point in W_n by as many edges as there are mentions of β in $\sigma(b)$. The point in W_n is connected to vertices in V_n by ordered edges determined by the ordered appearance of symbols in β. This new diagram is equivalent to the old one and, although not strictly stationary, has two graphical units which alternate.

Now telescope to the W levels and produce an equivalent diagram which has single points at each level and which is stationary, although it may have a multiple edge of the wrong multiplicity between level 0 and level 1. This is precisely case (a) in the statment of the theorem.

So, rejoining the main line of the argument with a diagram, J, which is proper, minimal and stationary and which has single edges between level 0 and level 1, assume that the substitution obtained from the diagram gives an aperiodic sequence. The argument proceeds to make equivalent modifications to the diagram in order to make the substitution 1-1 while keeping the edges between levels 0 and 1 single.

Suppose that $\sigma(a) = \sigma(b)$, $a \neq b \in \Lambda$. Each level $n \geq 1$ of vertices in I has a point (n, a) corresponding to a and similarly for b. Create a new sequence of layers of vertices in which $V'_n = \{(n, a')\} \cup V_n \setminus \{(n, a), (n, b)\}$ for all $n \geq 1$, where a' is some new symbol. Thus $|V'_n| = |V_n| - 1$.

To define E'_n, $n \geq 1$: For $c \neq a'$ there are as many edges with range $(n+1, c)$ as there were before and their source and ordinality is as before but for a natural coalition of the old (n, a) and (n, b) to (n, a'). The edges with range $(n+1, a')$ have source and ordinality exactly as with those with range (n, a) in the old diagram but with the same coalition as above. The edges in E'_0 are all single.

The claim is that this new ordered diagram $J' = (V'_n, E'_n)$ is order equivalent to the old one. To this end, form a third ordered diagram which will be intermediate to these two. Levels: $W_{2n} \equiv V_n$, $n \geq 0$ and $W_{2n-1} \equiv V'_n$, $n \geq 1$. Edges: $(2n, c)$, $n \geq 1$, is connected to $(2n-1, d)$ if and only if ($d \neq a'$ and $d = c$) or ($d = a'$ and $c = a$ or b) — no need for ordering of edges here. $(2n+1, c)$, $n \geq 0$, is connected to $(2n, d)$ with the same number of edges and same ordinality as those from c (if $c \neq a'$) or a (if $c = a'$) in the old diagram.

Telescoping this third diagram to odd levels produces (V'_n, E'_n) and a telescoping to even levels gives (V_n, E_n); hence the order equivalence as required.

Clearly J' is proper minimal etc. and it has single edges between levels 0 and 1.

J' defines a substitution which still may be not 1-1. In this case, if $|V'_n| \neq 1$ the procedure above can be repeated until the diagram is reduced either to singleton vertices at each level, and hence case (a) in the Theorem, or to a diagram whose substitution is 1-1 which, by arguments before, produces a primitive substitution and case (b) as required.

Equicontinuity and expansiveness are disjoint properties (see [W]). In this context, expansiveness holds whenever the system in question is a subshift. So the second part of the theorem is just a rephrasing of the first part.

In some sense, this is a converse to Theorem 5 and provides a complete characterisation of substitution minimal systems in terms of proper stationary ordered Bratteli diagrams.

Concluding remarks

The well-known Chacon example, where $\sigma(0) = 0010$ and $\sigma(1) = 1$, is an important case which is not covered by the analysis before — one of the substitutions has length 1. Indeed, there are complications at every turn since the improper Bratteli diagram generated by this substitution has a path which is both maximal and minimal as well as another maximal path distinct from another minimal path. Thus the path-sequence space has a fixed point as well as a copy of $X(\omega)$ and so it has no unique minimal subset.

However, it is straightforward to define primitive substitutions (those developed from the paired letter substitutions of [Q] for example) whose shift systems are conjugate to the Chacon system. With this adaption, the Chacon example is amenable to the analysis of this paper. In general, such an adaption produces a substitution sequence whose improper diagram has one maximal path and many minimal paths (or vice versa). In this case $\nu = 0$ and the problem of computation reduces to that of determining Q for the improper diagram of the paired letter substitution. This construction is pursued in detail in [H].

The author has developed a computer programme, in Q-Basic and suitable for a small Macintosh Computer, which computes the ordered repeating unit for the Proper Bratteli diagram for a given substitution system. The computations using Theorem 13 in examples (b), (c) and (d) above have been confirmed by the results of this programme.

The programme computes that K_0 for the Chacon system above is equal to $\mathbb{Z}[1/3] \oplus \mathbb{Z}$.

ACKNOWLEDGEMENT: The author is indebted to Christian Skau of the University of Trondheim whose hospitality and helpful conversations and communications were essential to the formation of this paper. Also he is grateful to Boris Solomyak, University of Washington, Seattle, for several references and to Bernard Host for helpful conversations and access to his unpublished course notes.

The author holds a William Gordon Seggie Brown Fellowship at the University of Edinburgh, and he wishes to thank the Seggie Brown Fund for its generous support while he wrote this paper. Part of this work was completed while enjoying the hospitality of the University of Trondheim with funding from the Norwegian Research Council for Science and Humanities, and for this help the author is most grateful.

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